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Shaping of System Responses with Minimax Optimization in the Time Domain

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A minimax problem is introduced for the terminal control of a generic dynamical system without disturbances. The maximum magnitude of the weighted output of the system is minimized over a finite interval by the control input of a prescribed class. Such important characteristics of the controlled system appear explicitly in the proposed problem as the maximum magnitude and settling property of the output. Two numerical examples are shown to illustrate the problem. A slewing experiment is also presented to demonstrate the application of the minimax optimal control.

I. Introduction

MANY efforts have been made to develop optimal control problems with quadratic criteria, especially for linear systems, over the past many years. Linear control system design with quadratic criteria (the LQ method) has such an advantage that the optimal control is analytically obtained as linear state feedback or a dynamic compensator. On the other hand, there are difficulties in the selection of free design parameters in a control synthesis based on the LQ method. Although some guidelines exist for weight selection,[1,2] and the exponential decay rate of the closed-loop state can be prescribed[2,3] in the LQ method, the free parameters affect control performance in an indirect and complicated manner. Iterative adjustment of the free parameters is a laborious task for a control designer, and a computational algorithm such as the one presented in Refs. 4 and 5 may be required to satisfy design specifications. It is often the case that a criterion different from quadratic criteria is preferable for terminal control in which open-loop control is acceptable. A minimum-fuel or minimum-time problem provides the perfect criterion when the specific design objective is to minimize fuel or time. The number of free parameters is confined in such cases, and laborious adjustment may not be required. To achieve good control performance with optimal control, it is necessary to analyze the control objective and to define a meaningful performance index so that the optimization yields satisfactory control behavior for the objective. In this paper, the performance of terminal control is measured in terms of the maximum magnitude of a controlled output vector and decay rate of the output in order to evaluate the time responses of the output more directly than quadratic criteria. A minimax problem in the time domain is introduced to shape time responses by minimizing the maximum magnitude of the weighted output with the control input of a prescribed class. The weighting matrix of the output vector is dependent on time to prescribe the decay rate of the output.

The minimax problem for terminal control is often called the Chebyshev minimax optimal control problem and has been studied for many years. Johnson[6] discussed the geometric properties of the minimax solution. Barry[7] gave a Mayer-type formulation of the problem and proposed an approximation method to yield a suboptimal minimax control that is arbitrarily close and in many cases identical to optimal control. Michael[8] demonstrated a general method for the efficient computation of minimax optimal control for nonlinear systems. In Refs. 9–12, the minimax problem is formulated as a state variable inequality constraint with a parameter to be optimized, and the necessary conditions of optimal control are derived. Although the formulation and solution of the minimax problem have been intensively studied in previous work, its application is restricted mainly to aerospace trajectory optimization[8,11–13] rather than the shaping of time responses of a generic dynamical system.

Minimax problems have also been studied since the early 1980s for linear feedback system design. The most popular minimax problem is H_∞ control theory.[14–17] A typical application of H_∞ control theory is the frequency shaping of singular-value Bode plots[18,19] that qualify the robustness, sensitivity, and disturbance rejection of a system. Whereas H_∞ control theory specifies feedback properties in the frequency domain, L_1 control theory[20–23] minimizes the maximum amplitude of system error when the disturbance to the system is unknown but bounded in amplitude. The set theoretic control synthesis technique[24,25] maximizes the amplitude of the disturbance that the system can tolerate without violating certain predetermined constraints and can be formulated as an application of L_1 optimal control theory.[26] Reference 27 proposes a design technique that is able to deal with performance specifications of the minimax type both in the frequency and time domain. Although these theories provide powerful tools for designing linear feedback control systems, they are mainly intended to reject disturbances and treat only linear time-invariant systems. The minimax problem in this paper treats terminal control of a generic dynamical system that can be nonlinear and/or time-variant. The objective of the optimization is to transfer a system to a desired state moderately with a prescribed decay rate rather than to reject disturbances. Reference 28 contains some preliminary results on the minimax problem for the shaping of time responses.

The next section discusses the application of the minimax problem to the shaping of system responses. In Sec. III, two numerical examples illustrate the responses of systems due to optimal control input and demonstrate control performance attained with minimax criteria. Section IV reports the results of a slewing experiment to verify that minimax optimal control can be satisfactorily realized in hardware.

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II. Shaping of System Responses

We treat a generic dynamical system with a state vector governed by an ordinary differential equation that may be nonlinear and/or time-variant. Such a dynamical system is expressed in the following form:

$$\dot{x}(t) = f\left[x(t), u(t), t\right]; \qquad x(t_0) = x_0 \tag{1}$$

$$y(t) = g[x(t), t] \tag{2}$$

where $x \in \boldsymbol{R}^n$ denotes the state vector, $u \in \Omega \subset \boldsymbol{R}^m$ the control input, $y \in \boldsymbol{R}^r$ the output vector to be controlled, t_0 the given initial time, and x_0 the given initial state. The family of admissible control is denoted by Ω. The system considered is deterministic, and disturbances to it are assumed not to exist. The output of the system may be a deviation of the system from a desired state, or a tracking error. System responses can be shaped by minimizing the maximum value of a vector norm of the output vector. Since minimization of the maximum magnitude of the output does not guarantee convergence of the output to zero, the output should be weighted by a time-dependent weighting matrix that increases as time increases when convergence is required. Therefore, the performance index to shape system responses is given as

$$J = \max_{t_0 \le t \le t_f} \| W(t) g[x(t), t] \| \tag{3}$$

where the matrix $W(t)$ is a time-dependent weighting matrix to specify the settling property of the output and t_f the given terminal time. Some variations in the performance index and terminal conditions are possible with a slight modification. The vector norm is assumed to satisfy the inequality

$$|x_i| \le \|x\|; \qquad x = (x_i) \in \boldsymbol{R}^n \tag{4}$$

If the weighting matrix $W(t)$ is a diagonal matrix

$$W(t) = \operatorname{diag}(w_1 e^{\beta_1 t}, \ldots, w_r e^{\beta_r t}), \quad (w_i > 0; \quad i = 1, 2, \ldots, r) \tag{5}$$

then the magnitudes of the elements of the output vector are bounded as follows:

$$|y_i(t)| \le J e^{-\beta_i t} / w_i \tag{6}$$

The system responses are explicitly predicted from design parameters w_i and β_i, and from the value of the performance index as shown in Eq. (6). The parameters w_i are scaling factors to evaluate all output in the same unit, and the parameters β_i specify the convergent property of the output. An exponential weight is often employed in the LQ method to prescribe degree of stability.[2] The exponential weight in the minimax problem is essential for the convergence of output, in contrast to the LQ case in which convergence of output is guaranteed without it. Although other form of weight than the exponential function is possible, modification of the weighting matrix is beyond the scope of this paper.

The following guideline may be reasonable for the selection of design parameters: Choose the parameters β_i by taking desirable settling time into account, e.g.,

$$e^{-\beta_i \sigma_i} = 0.05 \tag{7}$$

and the scaling factors w_i so that

$$w_1 : w_2 : \cdots : w_r = |y_1|_{\max} : |y_2|_{\max} : \cdots : |y_r|_{\max} \tag{8}$$

where $|y_i|_{\max}$ denotes the acceptable value of $|y_i(t)|$ over the control interval, and σ_i the desirable settling time of the output y_i in the range within $\pm 5\%$ of $|y_i|_{\max}$. The value of a scaling factor itself does not play any important role in the shaping of system responses. The ratio between the scaling factors affects controlled responses. If design specifications are not satisfied in the first trial, the design parameters should be adjusted through iteration, as is similar to the cases of quadratic criteria. However, the iterative selection proceeds more readily than the quadratic criteria cases because of the explicit appearance of design parameters in the control performance, Eq. (6).

An arbitrary vector norm that satisfies Eq. (4) may be selected in order to evaluate the magnitude of the output vector. This paper employs l_{2p} norm ($p = 1, 2, 3, \ldots < \infty$) on a finite dimensional Euclidean space as the vector norm. The l_p norm $\|x\|_p$ of a vector $x = (x_i) \in \boldsymbol{R}^n$ is defined as[29]

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \tag{9}$$

for finite p and

$$\|x\|_\infty = \max_i |x_i| \tag{10}$$

for $p = \infty$. Equation (4) is satisfied by the l_p norm. The l_2 norm is also termed the Euclidean norm. The l_∞ norm, simply defined as the maximum absolute value of the elements of a vector, is preferable in the shaping of system responses because it does not overestimate the upper bound of the magnitude of each output channel. For a finite p, a vector does not necessarily have an element whose absolute value equals the l_p norm of the vector, as illustrated in Fig. 1 for a two-dimensional case. However, since an inequality

$$\|x\|_\infty \le \|x\|_p \le n^{1/p} \|x\|_\infty \tag{11}$$

holds, evaluation by the l_p norm differs negligibly from the l_∞ norm when $n^{1/p}$ is close enough to 1. The two-point boundary-value problem (TPBVP) is presented in the following discussion for general l_{2p} norm ($p = 1, 2, 3, \ldots < \infty$). This paper treats only the l_{2p} norm ($p = 1, 2, 3, \ldots < \infty$) because of the smoothness condition required for the direct application of the standard first-order necessary conditions. The derivative of the l_p norm with respect to its argument is not continuous for odd p and infinite p.

We compute the solution of the minimax problem with the procedure presented in Ref. 8. By invoking a well-known theorem in functional analysis, the performance index, Eq. (3), can be approximated well by the following integral performance index with enough large q:

$$J_q = \left(\int_{t_0}^{t_f} \| Wg \|_{2p}^q \, \mathrm{d}t \right)^{1/q} \tag{12}$$

where the l_{2p} norm is chosen as the vector norm because of the reason just stated. The basis for introducing the performance index, Eq. (12), is that the performance index, Eq. (3), is identical to $\lim_{q \to \infty} J_q$. By introducing a new state variable $z_q(t)$, defined as

$$z_q(t) = \left(\int_{t_0}^{t} \| Wg \|_{2p}^q \, \mathrm{d}\tau \right)^{1/q} \tag{13}$$

the problem is transformed to the Mayer type

$$J_q = z_q(t_f) \tag{14}$$

The new state variable z_q is governed by the differential equation

$$\dot{z}_q = \frac{\| Wg \|_{2p}^q}{q z_q^{q-1}} \tag{15}$$

The Hamiltonian H for this case is defined as

$$H = \psi^T f + \phi \frac{\| Wg \|_{2p}^q}{q z_q^{q-1}} \tag{16}$$

where the vector $\psi \in \boldsymbol{R}^n$ and scalar ϕ are costate variables that correspond to x and z_q, respectively. The minimum principle[1] implies that the optimal control input must minimize the Hamiltonian along the optimal trajectory, and this condition yields the optimal control input

$$u(t) = \arg[\min_{u \in \Omega} H] \tag{17}$$

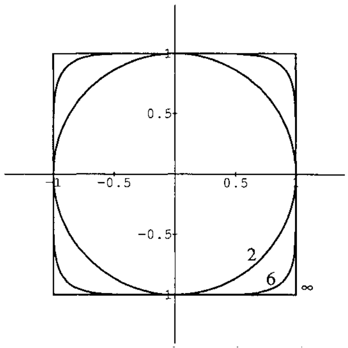

Fig. 1 Unit circles: $\|x\|_p = 1$ ($p = 2, 6, \infty$).

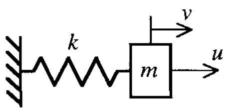

Fig. 2 Mass-spring system.

The costate variables are governed by the following Euler-Lagrange equations:

$$\dot{\psi} = -H_x = -f_x^T\psi - \phi\,\frac{\|Wg\|_{2p}^{q-2p}}{z_q^{q-1}}\,(Wg_x)^T\big[(Wg).\wedge(2p-1)\big] \tag{18}$$

$$\dot{\phi} = -H_{z_q} = \phi\,\frac{(q-1)\|Wg\|_{2p}^{q}}{q z_q^{q}} \tag{19}$$

with the transversality conditions

$$\psi(t_f) = \frac{\partial z_q(t_f)}{\partial x(t_f)} = 0 \tag{20}$$

$$\phi(t_f) = \frac{\partial z_q(t_f)}{\partial z_q(t_f)} = 1 \tag{21}$$

where $x\,.\wedge p$ represents a vector that consists of x_i to p power. The TPBVP is given by Eqs. (1), (15), and (18–21) with the optimal control input determined by Eq. (17).

It is straightforward to show from Eqs. (14), (15), (19), and (21) that

$$\frac{\dot{\phi}}{\phi} = (q-1)\,\frac{\dot{z}_q}{z_q} \tag{22}$$

and the costate ϕ is expressed explicitly as

$$\phi(t) = \left[\frac{z_q(t)}{J_q}\right]^{q-1} \tag{23}$$

Then the Hamiltonian and Euler-Lagrange equation are rearranged as

$$H = \psi^T f + \frac{\|Wg\|_{2p}^{q}}{q J_q^{q-1}} \tag{24}$$

$$\dot{\psi} = -f_x^T\psi - \frac{\|Wg\|_{2p}^{q-2p}}{J_q^{q-1}}\,(Wg_x)^T\big[(Wg).\wedge(2p-1)\big] \tag{25}$$

The state z_q and costate ϕ do not appear in Eqs. (24) and (25).

An alternative to the time-dependent weighting matrix for the convergence of output is a constraint on the terminal state. When a terminal constraint is introduced to the optimization problem, only the terminal condition of the TPBVP requires modification. If the terminal state is constrained to satisfy the equation

$$\xi\big[x(t_f), t_f\big] = 0, \qquad (\xi \in \boldsymbol{R}^{nc}) \tag{26}$$

then the terminal condition of the costate ψ is modified as

$$\psi(t_f) = \xi_x^T\big[x(t_f), t_f\big]\nu \tag{27}$$

where $\nu \in \boldsymbol{R}^{nc}$ is a Lagrange multiplier to be determined. The other conditions are not affected by the terminal constraint.

III. Numerical Examples

Numerical Method for Solution

Two numerical examples are presented to demonstrate how the responses of controlled systems are shaped by the proposed minimax problem. The clipping-off conjugate gradient algorithm[30] is employed as the numerical method to solve the optimization problem. Given an estimate u_i of the optimal control, the algorithm carries out the linear search of a parameter α_i such that

$$\min_{\alpha_i \geq 0} J[u_i + \alpha_i s_i] \tag{28}$$

where s_i is determined from the Hamiltonian and other variables by an algorithm similar to the usual conjugate gradient method, and the magnitude of $u_i + \alpha_i s_i$ is clipped off not to violate the constraint on the magnitude of the control input. The control input u_i is replaced by $u_{i+1} = u_i + \alpha_i s_i$, and the foregoing processes are iterated.

Mass-Spring System

The first example is a mass-spring system of the second order as depicted in Fig. 2. A single control input u is applied to the mass. The state vector x of the system consists of the displacement v of the mass and its velocity, and the controlled output y is the displacement, i.e.,

$$x = [v \quad \dot{v}]^T, \qquad y = v \tag{29}$$

The initial condition x_0, initial time t_0, terminal time t_f, mass m, and spring constant k are selected as

$$x_0 = [1 \quad 0]^T, \quad t_0 = 0, \quad t_f = 10, \quad m = 1, \quad k = 1 \tag{30}$$

The time-dependent weight W is chosen as follows for this system:

$$W = e^{0.5t} \tag{31}$$

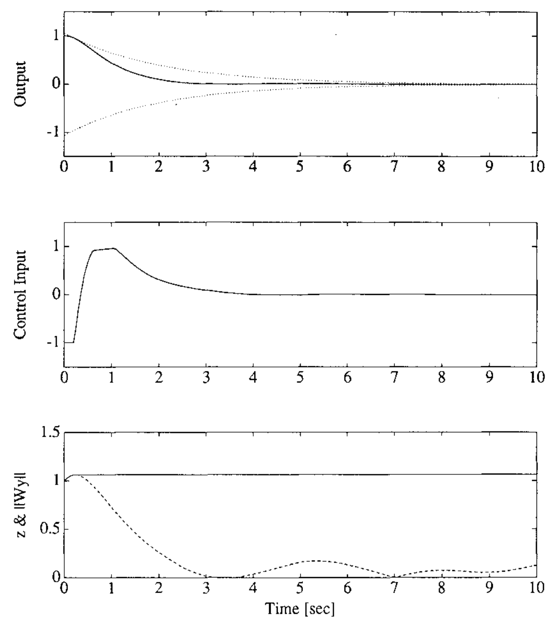

Fig. 3 Optimal solution for mass-spring system with minimax criterion: unconstrained terminal state.

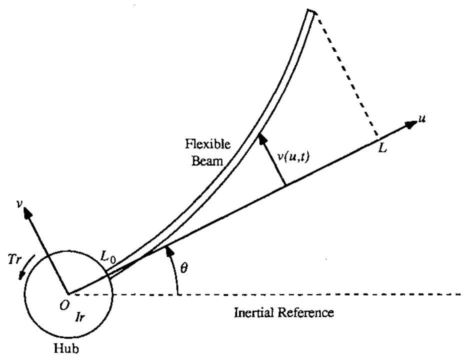

Fig. 4 Slew maneuver model.

which is expected to achieve the settling time of 6.0 s. The norm of the weighted output is identical to the absolute value. The family of admissible control Ω in this case is

$$\Omega = \{u : |u| \leq 1\} \tag{32}$$

The optimized time history of variables in the mass-spring system is shown in Fig. 3. The figure indicates the controlled output $y(t)$, control input $u(t)$, weighted norm $\|W(t)y(t)\|$ (broken line), and $z(t)$ (solid line), defined as

$$z(t) = \max_{t_0 \leq \tau \leq t} \|W(\tau)y(\tau)\| \tag{33}$$

The bounds of the output magnitude

$$y_b(t) = \pm z(t_f)\exp(-0.5t)$$

(dotted lines) are overlapped to the time history of the output variable. The resulting value of the performance index is equal to 1.06, which is 6% larger than $z(t_0)$. The magnitude of the output is bounded by the exponential function as expected. The attained settling time is 2.4 s, which is much smaller than expected. A bang-bang control $u = -1$ is seen only until 0.22 s. It is also shown in the figure that $z = \|Wy\|$ holds in [0,0.22] and $z < \|Wy\|$ in (0.22,10].

Slew Maneuver

The second example treats the slew maneuver of a rigid body with a flexible appendage. The slew maneuver of such a system has received much attention in the field of control of flexible space structures.[31-35] The control objective is to change the attitude angle of the rigid body with minimal vibration excitation of the flexible appendage. It has been pointed out[35] that the simple constant-gain feedback control of a large-angle maneuver often results in poor vibration suppression of the flexible appendage. Tracking-type feedback control is effective for carrying out both large-angle maneuver and terminal pointing/vibration suppression, and open-loop control should be designed to determine the reference motion of the slew maneuver to be tracked. The present system is modeled as the two-dimensional rotary motion of a rigid hub equipped with a flexible beam as illustrated in Fig. 4. The rigid hub is actuated in the plane by the control torque T_r about the center of rotation O, and the flexible beam has deflection $v(u,t)$ in the same plane. The flexible beam is assumed to be Bernoulli-Euler beam, with one end ($u = L_0$) fixed on the hub and the other end ($u = L$) free. The output variables to be controlled are the hub angle θ and the bending moment at the root of the beam M_0. The parameters of the model are adopted from the experimental model in the next section. A state equation is constructed first with the 16 cantilever modes, i.e., constrained modes[36] of the Bernoulli-Euler beam by the same procedure as in Ref. 33, then the unconstrained modes are obtained by solving an eigenvalue problem[37] and are truncated by the second flexible mode. The output vector y is defined as

$$y = [\theta \quad M_0]^T \tag{34}$$

Note that the derivatives of θ and M_0 with respect to time are not included in the controlled output. Because the objective of the optimization is the shaping of output responses and not of their time derivatives, one does not have to be concerned with how large the time derivatives are. The hub angle is equal to -1.05 rad (-60 deg) and the flexible beam is at equilibrium at the initial time. The output is controlled to approach zero during a time interval $[0, t_f]$ ($t_f = 5, 10$). The family of admissible control Ω is given as

$$\Omega = \{T_r : |T_r| \leq 0.15\} \tag{35}$$

The time-dependent weighting matrix W is chosen for this example as

$$W = e^{\beta t}\begin{bmatrix} 1 & 0 \\ 0 & 100 \end{bmatrix} \tag{36}$$

where the parameter β is equal to 0.5 when the terminal state is not constrained and 0 when it is constrained. The scaling factor for the bending moment is selected to balance the values

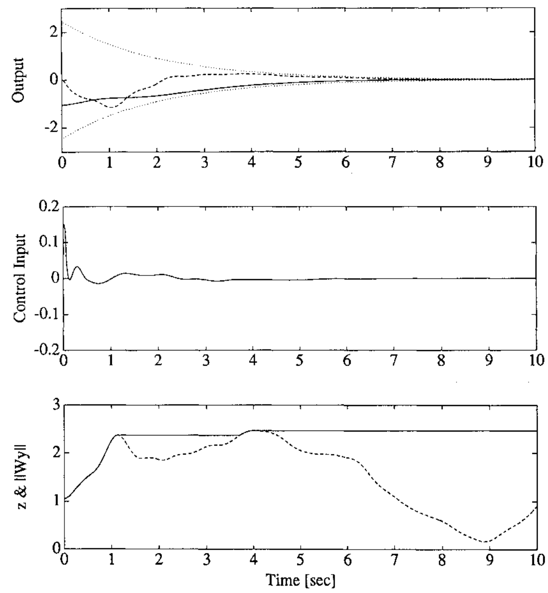

Fig. 5 Optimal solution for slew maneuver with quadratic criterion: $\beta = 0.5$, unconstrained terminal state.

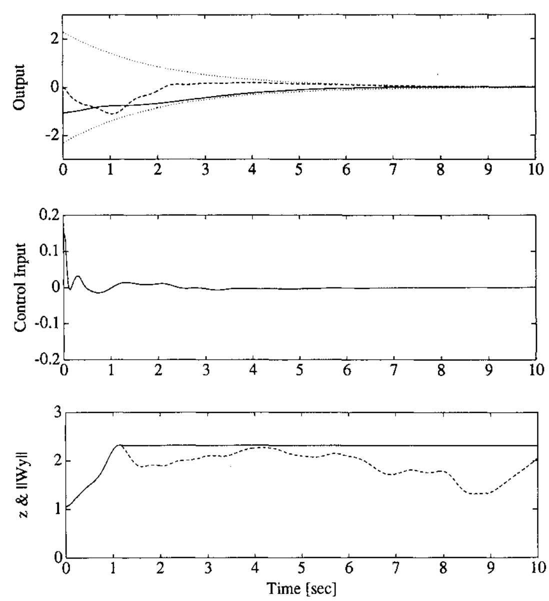

Fig. 6 Optimal solution for slew maneuver with minimax criterion: $\beta = 0.5$, l_2 norm, unconstrained terminal state.

of two output variables by consulting the maximum allowable bending moment of the beam, 0.02 N.m.

In order to compare with the solutions of the minimax problem, numerical solutions are calculated for the problem of a quadratic criterion with a performance index of the form

$$J = \left\{ \int_{t_0}^{t_f} g^T W^T W g \, \mathrm{d}t \right\}^{\frac{1}{2}} \tag{37}$$

Figure 5 indicates the solutions: the attitude angle $\theta(t)$ (solid line), root moment of the beam $M_0(t)$ scaled up by 100 (broken line), control input $T_r(t)$, weighted norm $\| W(t)y(t)\|$ (broken line), and $z(t)$ (solid line) defined as in Eq. (33). The vector norm is the l_2 norm, and the bounds of the output magnitude $y_b(t) = \pm z(t_f)\exp(-\beta t)$ (dotted lines) are overlapped on the time history of the output variables. The resulting value of $z(t_f)$ is equal to 2.47. Numerical solutions are presented in Figs. 6 and 7 for the problems of minimax criteria with the l_2 and l_{80} norms, respectively. The parameter q for the approximate solutions is taken to be 16 for the l_2 norm case and 8 for the l_{80} norm case, respectively. The resulting values of $z(t_f)$ are, respectively, 2.31 for the l_2 norm case and 1.97 for the l_{80} norm case. Values of the approximate performance indices are 3 and 13.7% larger than $z(t_f)$ for the respective case. Only slight differences are observed between the output responses of the three cases. Satisfactory output responses are attainable in this example with either the quadratic criterion or minimax criteria by employing the time-dependent weighting matrix introduced in the preceding section. However, the peak of the weighted norm at 4 s in Fig. 5 is suppressed in the result of Fig. 6, and this observation demonstrates that output responses are shaped more effectively with the use of minimax rather than quadratic criteria. The reduction in the value of the performance index amounts to 15% by modifying the vector norm to the l_{80} norm from the l_2 norm. This reduction is attributed mainly to the fact that the l_{80} norm does not overestimate the upper bound of the magnitude of the output vector, as mentioned in the preceding section. The l_2 norm can be 41% larger than the maximum absolute value of the elements of the weighted output, since $2^{1/2} = 1.41$. In contrast, overestimation by the l_{80} norm is no more than 1%, since $2^{1/80} = 1.01$. Figure 8 shows the solution for the minimax criterion, with the terminal state constrained as zero. The vector norm is the l_{80} norm. The resulting value of the performance index is 1.27; it is identical to the maximum absolute value of the root moment $M_0(t)$ multiplied by the scaling factor, because the weighting matrix does not depend on time in this case.

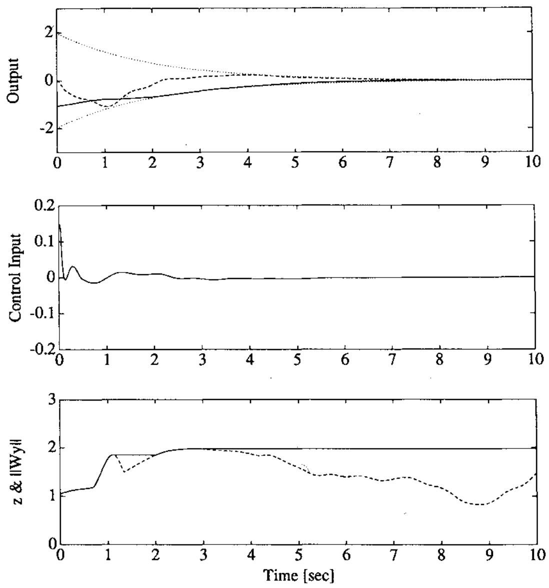

Fig. 7 Optimal solution for slew maneuver with minimax criterion: $\beta = 0.5$, l_{80} norm, unconstrained terminal state.

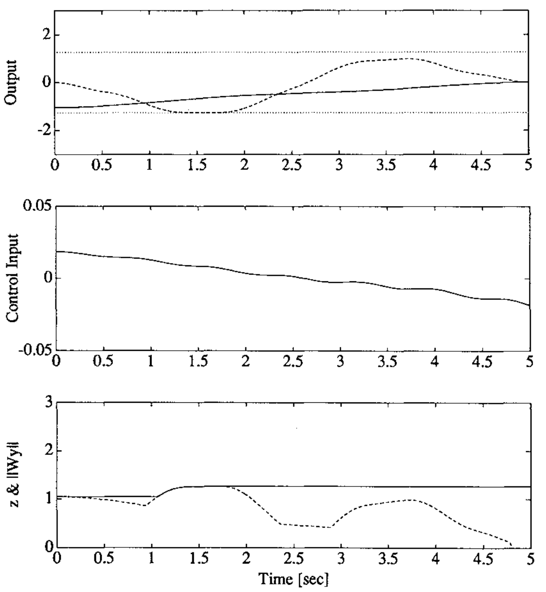

Fig. 8 Optimal solution for slew maneuver with minimax criterion: $\beta = 0$, l_{80} norm, constrained terminal state.

Table 1 Model parameters

Flexible beam	
Length	1.155 m
Thickness	5.0×10^{-4} m
Height	2.0×10^{-2} m
Bending rigidity	1.48×10^{-2} N·m^2
Mass per unit length	6.39×10^{-2} kg/m
Air drag	1.1×10^{-2} N·s/m^2
Kelvin-Voigt damping	5×10^{-5} N·m^2·s
Rigid hub	
Radius	3.38×10^{-2} m
Moment of inertia	3.43×10^{-2} kg·m^2
Viscous damping	1.5×10^{-4} N·m·s/rad
Support board	
Diameter	1.6×10^{-1} m

IV. Slewing Experiment

Hardware Setup

A slewing experiment is performed to verify that the minimax optimal control can be realized satisfactorily in an actual environment. The experiment setup is described in detail in Ref. 31 and is depicted here briefly, although including modifications made after the experiment in Ref. 31. The parameters of the experimental model are listed in Table 1. The control torque is given by a dc torque motor, and sensing is provided by a tachometer at the rotary shaft of the motor and four strain gauges located at the beam root. The output of the tachometer is integrated by an analog circuit to obtain the attitude angle of the hub. Those sensors are necessary for the tracking control law introduced here. The model is set on a zero-G simulation table to cancel the influence of the Earth's gravity on the flexible beam. The model beam is equipped with three support boards to receive the air flow from the surface of the table. A digital computer processes sensor data and sends out the control signal in the sample interval of 10 ms. The control software is coded in C language.

Tracking Control Law

Although the minimax problem results in an open-loop control, a tracking control law is employed in order to compensate for modeling error and disturbances that are unavoidable in the experiment. The tracking control law in this experiment was proposed in Ref. 35 for the near-minimum-time maneuvers of distributed parameter systems and is a modified version

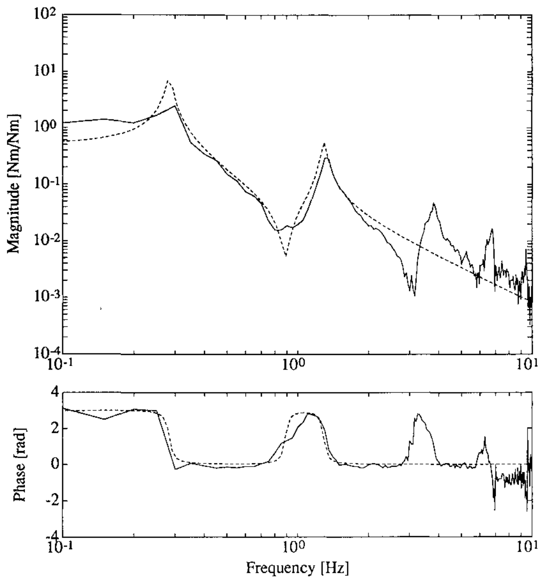

Fig. 9 Transfer function from control torque to bending moment at beam root (solid line: measured, broken line: state-space model).

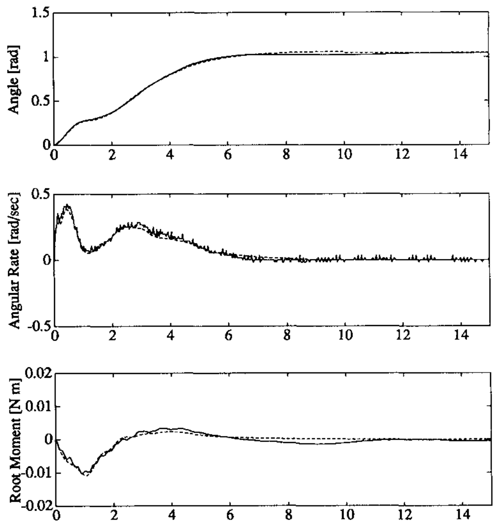

Fig. 10 Experimental results with minimax criterion: $\beta = 0.5$, l_{∞} norm, unconstrained terminal state (solid line: measured, broken line: reference).

of the control algorithm presented in Refs. 31 and 34. The control torque T_r is determined by the following algorithm:

$$T_r = T_{r\text{ref}} - \frac{1}{a_1}\Big\{a_2(\theta - \theta_{\text{ref}}) + k(\dot{\theta} - \dot{\theta}_{\text{ref}}) + (b - a_1)\big[(l_0 S_0 - M_0) - (l_0 S_0 - M_0)_{\text{ref}}\big]\Big\} \quad (38)$$

where a_1, a_2, b, and k are positive constants, M_0 and S_0 denote bending moment and shear force at the beam root, respectively, and $(\ )_{\text{ref}}$ denotes the reference to be tracked. The tracking control law, Eq. (38), guarantees convergence of the trajectory of the system to the reference trajectory, if the reference trajectory is an exact solution of the partial differential equation governing the slew maneuver. In this experiment, the reference trajectories are calculated from the solutions of optimization problems for the state-space model truncated by two flexible modes and are not an exact solution of the original equation of motion. However, it is proved in the experiment that the tracking control law works satisfactorily.

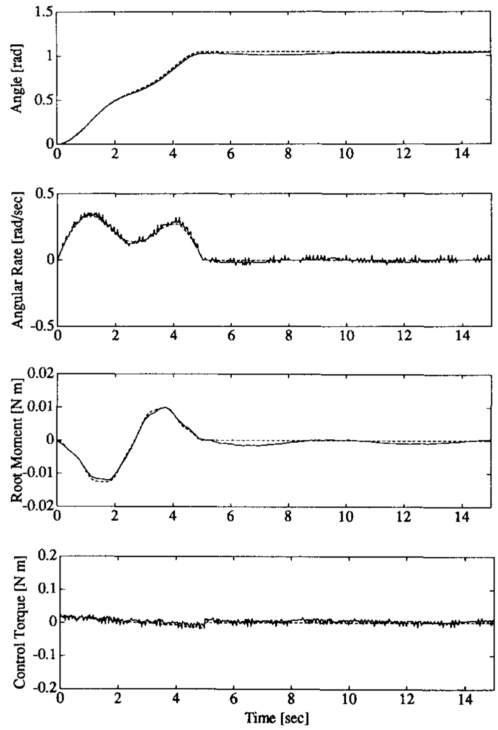

Fig. 11 Experimental results with minimax criterion: $\beta = 0$, l_{∞} norm, constrained terminal state (solid line: measured, broken line: reference).

Experimental Results

Measured and predicted transfer functions from the control torque T_r to bending moment at the beam root M_0 are compared in Fig. 9. The mass per unit length of the beam is adjusted so that the influence of the support boards is compensated for and the responses of the mathematical model agree with the measured ones in terms of both the time and frequency domains. The resulting value of the mass per unit length of the beam is 130% larger than the nominal one. Five flexible modes are observed below 10 Hz in the measured transfer function, whereas the state-space model includes only two such flexible modes. The residual modes are distinguished because of the log-scale in the Bode plot and are not dominant in the time response.

The tracking control law, Eq. (38), is conducted for 15 s with $a_1 = 100$, $a_2 = 30$ (N.m), $b = 1000$, and $k = 40$ (N.m·s). The reference trajectories are calculated from the optimal solutions in the preceding section so as to rotate the attitude angle from 0 to 1.05 rad and are embedded as $\theta_{\text{ref}} = 1.05$ (rad), $M_{0\,\text{ref}} = 0$, and $S_{0\,\text{ref}} = 0$ after the terminal time t_f of each optimization problem. The tracking control law is identical to the mission function control[31,34] after the terminal time.

Experimental results are shown in Fig. 10 for an unconstrained problem of the minimax criterion with the l_{∞} norm. The measured responses agree well with the reference responses and are deemed satisfactory. Experimental results are shown in Fig. 11 for the minimax criterion, with the terminal state constrained as zero. The experimental results correspond to the numerical example in Fig. 8 in which the l_{∞} norm is employed as the vector norm. Satisfactory responses are attained in the experiment.

V. Conclusions

This paper discusses the shaping of time responses of a generic dynamical system. A minimax problem is introduced in order to shape time responses through an optimal control problem. The main purpose of the proposed minimax problem is to determine a control input minimizing the maximum magnitude of a weighted output of a dynamical system over a finite time interval. The dynamical system may be nonlinear and/or time-variant. Disturbances to the system are assumed not to exist. Through the use of a time-dependent weighting matrix on the ouput, such important characteristics of the controlled system appear explicitly in the proposed problem as the maximum magnitude and settling property of the output. This is to say that the controlled responses can be predicted without recourse to graphic representation, and the adjustment of design parameters proceeds more readily than in the cases of quadratic criteria. The l_{2p} norm is employed as the vector norm to evaluate the magnitude of the weighted output vector. The minimax problem is transformed to a problem with an integral performance index and solved by a gradient method. Numerical examples are presented in order to demonstrate how the responses of controlled systems are shaped with the minimax criteria. A slewing experiment demonstrates that minimax optimal control can be realized satisfactorily in an actual environment.

Further research on the minimax problem is recommended, especially to explore its analytical features. One potential focus for such analysis is the minimax problem for linear systems. The minimax problem can be formulated as an approximation problem in a normed vector space, which is an application of functional analysis. Once the infimum of the performance index of the minimax problem is obtained by solving a dual problem, the problem reduces to an optimization problem with a state-variable inequality constraint and may be solved by the numerical methods developed in the past.

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